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Nonlinear Faraday resonance

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A cylinder containing liquid with a free surface is subjected to a vertical oscillation of amplitude $\epsilon g/\omega^2$ and frequency 2ω, where ω is within $O(\epsilon\omega)$ of the natural frequency of a particular (*primary*) mode in the surface-wave spectrum and $0 < \epsilon \ll 1$. A Lagrangian formulation, which includes terms of second and fourth order in the primary mode and second order in the secondary modes (which are excited by the primary mode), together with the method of averaging, leads to a Hamiltonian system for the slowly varying amplitudes of the primary mode. The fixed points (which correspond to harmonic motions) and phase-plane trajectories and their perturbations due to linear damping are determined. It is shown that $\epsilon > \delta$, where δ is the damping ratio (actual/critical) of the primary mode, is a necessary condition for subharmonic response of that mode. Explicit results are given for the dominant axisymmetric and antisymmetric modes in a circular cylinder. Internal resonance, in which a pair of modes have frequencies that approximate ω and 2ω, is discussed separately, and the fixed points and their stability for the special case $\omega_2 = 2\omega_1$ are determined. Internal resonance for $\omega_2 = \omega_1$ is discussed in an appendix.

1. Introduction

The subharmonic excitation of surface waves in a vertically oscillating basin, originally observed by Faraday (1831), has been analysed by Rayleigh (1883*a*,*b*), Benjamin & Ursell (1954), Dodge, Kana & Abramson (1965), Ockendon & Ockendon (1973) and Henstock & Sani (1974). Benjamin & Ursell reduced the description of small disturbances to Mathieu's equation and invoked the known results for the stability of the solutions of that equation to support Rayleigh's conclusion that the primary oscillations of the free surface occur at half the frequency of the oscillation of the container.† Ockendon & Ockendon (1973) extended the analyses of Rayleigh and Benjamin & Ursell to small but finite amplitudes but did not calculate the parameter that measures the effects of nonlinearity. The present analysis provides an explicit result for this parameter and incorporates linear damping.

Dodge *et al.* (1965) have given a finite-amplitude analysis for a circular basin that should be equivalent to that developed here, but their equations of motion for the modal amplitudes violate reciprocity conditions that are implicit in the underlying (Newtonian) mechanics; see Appendix E. Henstock & Sani (1974) also have given a finite-amplitude analysis for a circular basin, but they applied the free-surface boundary conditions at the equilibrium, rather than the diplaced, position of the free

† Benjamin & Ursell's (1954) statement that their work was 'made possible by the development of Mathieu functions since Rayleigh's time' suggests that they may have overlooked Rayleigh's 1887 paper, in which he applies the theory of Hill's equation to subharmonic excitation, albeit not explicitly to Faraday's problem.

surface and obtained a correction to the resonant frequency that is first order in the amplitude (the true correction must be second order).

Keolian *et al.* (1981), Gollub & Meyer (1983) and Cilberto & Gollub (1984) have recently reported observations of strongly nonlinear motions in vertically oscillating basins, some of which involve either resonant coupling between several modes or chaotic instabilities or both. These strongly nonlinear motions presumably lie outside of the scope of the present investigation.

I begin my analysis, in §2, by expressing the motion of the free surface in terms of the corresponding normal modes and calculating the Lagrangian through terms of fourth order in the modal amplitudes on the provisional assumption that dissipation is negligible. This calculation is based on an earlier formulation for nonlinear gravity waves in a cylinder of arbitrary cross section (Miles 1976), equations from which are cited by the prefix I. Capillary effects, which dominate gravitational effects for sufficiently small wavelengths (as in Faraday's experiments), are considered in Appendix D.

Appealing to the results cited above, I assume that the natural frequency of the primary mode, say ω_1, approximates half the frequency of the vertical displacement of the basin,

$$z_0 = a_0 \cos 2\omega t \quad (\omega^2 |a_0| \ll g). \tag{1.1}$$

Quadratic nonlinearity implies the excitation of time-independent and second-harmonic components of both this primary mode and certain secondary modes. (Higher harmonics are excited through higher-order interactions, but are not significant in the present context.) I first assume that only one such primary mode is resonantly excited and that none of the natural frequencies of the secondary modes approximates 2ω (this excludes internal resonance, which is considered in §6). Nonlinearity also implies (except for special initial conditions) slow variations of the amplitudes and phases of the sinusoidal carriers, which, together with the preceding arguments, leads me to posit

$$\eta_n = \delta_{1n}(\hat{A} \cos \omega t + \hat{B} \sin \omega t) + \hat{A}_n \cos 2\omega t + \hat{B}_n \sin 2\omega t + \hat{C}_n \tag{1.2}$$

for the generalized coordinate of the nth mode, where δ_{1n} is the Kronecker delta, $\hat{A}$, $\hat{B}$, $\hat{A}_n$, $\hat{B}_n$ and $\hat{C}_n$ are slowly varying, dimensional amplitudes, and, by hypothesis, $\hat{A}_n$, $\hat{B}_n$ and $\hat{C}_n$ are of the order of $\hat{A}^2$ and $\hat{B}^2$.

I obtain the evolution equations for these slowly varying amplitudes in §3 by averaging the Lagrangian over a 2π interval of ωt and then invoking Hamilton's principle. The secondary amplitudes $\hat{A}_n, \ldots$ prove to be quasi-steady in the sense that their temporal derivatives are negligible in (the present approximations to) the evolution equations, by virtue of which they may be expressed as quadratic functions of $\hat{A}$ and $\hat{B}$. The elimination of these secondary amplitudes from the average Lagrangian then leads to a Hamiltonian system for $\hat{A}$ and $\hat{B}$ which is isomorphic to that for a simple pendulum that is subjected to a vertical oscillation of its point of support (see Appendix A). I establish the fixed points and phase-plane trajectories of this Hamiltonian system in §4.

I introduce weak, linear damping in §5, assuming the availability of the damping ratio δ (of actual to critical damping) for the primary mode. This parameter is perhaps best determined through the direct measurement of the decay of free oscillations in that mode, although theoretical calculations (Miles 1967) can provide reasonably accurate values of δ for a clean free surface in a hydrophilic basin. Every phase-plane trajectory for the damped system spirals into a stable fixed point (at which the amplitudes became constant) of the evolution equations. There may be one, two, or three such fixed points (one of which may be the null point of no relative motion), depending on the proximity of ω_1 and ω.

Only two parameters, which are dimensionless measures of damping and of the proximity of ω and ω_1, enter the normalized phase-plane equations; however, the determination of the actual displacements and other dynamical quantities from the solutions of these normalized equations requires a calculation that is equivalent to that of the frequency of free oscillations of small but finite amplitude in the primary mode (I (6.5)). Explicit results are available for two-dimensional waves in a rectangular basin (Tdjbakhsh & Keller 1960), the simplest (no nodal lines) three-dimensional mode in a rectangular basin (Verma & Keller 1962), the dominant axisymmetric mode (which has one nodal circle) in a circular cylinder (Mack 1962), and the dominant antisymmetric mode (which has one nodal diameter) in a circular cylinder (Miles 1984).

The development in §§1–5 excludes the possibility of internal resonance, which may occur if the frequencies of a pair of modes approximate ω and $n\omega$ ($n = 1, 2, 3, \ldots$). The coupling between two such modes decreases with increasing n, and it is only for $n = 1$ or 2 that internal resonance is likely to be observable.

The case of equal frequencies arises naturally in a circular cylinder, for which the non-axisymmetric modes occur in degenerate pairs; however, the coupled motion of such a pair necessarily comprises angular momentum, which cannot be generated by vertical excitation. It also is possible to have approximately coincidental eigenvalues in the doubly infinite, discrete spectrum for any basin (cf. Cilberto & Gollub 1984). I give the general formulation for this problem in Appendix C, but, discouraged by the algebraic complexity, have not obtained explicit results for specific cases.

Internal resonance with $\omega_2 = 2\omega_1$ for two gravity-wave modes with wavenumbers k_2 and k_1 in cylindrical basin of depth d requires

$$k_2 \tanh k_2 d = 4k_1 \tanh k_1 d, \tag{1.3}$$

which may be achieved by a unique choice of d for any pair of eigenvalues for which $2 < k_2/k_1 < 4$. The lowest such resonance for axisymmetric motion in a circular cylinder of radius a corresponds to the modes with one and three nodal circles and is given by (Mack 1962)

$$k_1 a = 3.8317, \quad k_2 a = 10.1735, \quad d/a = 0.1981. \tag{1.4}$$

The coupling coefficient for this pair proves to be rather small (see §6). An example with a much larger coupling coefficient is resonance between the dominant antisymmetric and axisymmetric modes, for which (Miles 1984)

$$k_1 a = 1.8412, \quad k_2 a = 3.8317, \quad d/a = 0.1523. \tag{1.5}$$

I consider the general case of 2:1 internal resonance in §6. The comprehension of two primary modes requires a four-dimensional phase space. I determine the fixed points in this space for the special case $\omega_2 = 2\omega_1$ (as contrasted with $\omega_2 - 2\omega_1 = O(\epsilon\omega_1)$) and zero damping. There then is at least one stable fixed point (there may be as many as five) for every value of $(\omega - \omega_1)/\epsilon\omega$, and there are no Hopf-bifurcation points, which suggests that, in the presence of small but finite damping, every solution will terminate either in the null configuration or in a stable, harmonic oscillation after transient motion from specified initial conditions and that neither periodic limit cycles nor chaotic motions will be realized.

2. The Lagrangian

We pose the free-surface displacement (relative to the plane of the level surface, which is moving with the basin) in the form

$$\eta(\boldsymbol{x}, t) = \eta_n(t)\,\psi_n(\boldsymbol{x}) \quad (\boldsymbol{x} \text{ in } S), \tag{2.1}$$

where repeated indices are summed over the participating modes, the η_n are generalized coordinates, the ψ_n are the eigenfunctions determined by

$$(\nabla^2 + k_n^2)\,\psi_n = 0 \quad (\boldsymbol{x} \text{ in } S), \quad \boldsymbol{n}\cdot\nabla\psi = 0 \quad \text{on } \partial S, \quad \iint \psi_m \psi_n \,\mathrm{d}S = \delta_{mn} S, \tag{2.2a,b,c}$$

k_n are the eigenvalues, S is the cross-section of the cylindrical basin, and δ_{mn} is the Kronecker delta. The corresponding Lagrangian, as given by I (3.6) after setting $q_n = \eta_n$, $Q_n = 0$ and $\dot{v} = \ddot{z}_0$ in I (3.5c), is given by

$$L \equiv (\rho S)^{-1}(T - V) = \tfrac{1}{2} a_{mn} \dot{\eta}_m \dot{\eta}_n - \tfrac{1}{2}(g + \ddot{z}_0)\,\eta_n \eta_n, \tag{2.3}$$

where ρ is the fluid density, T and V are the kinetic- and potential-energy densities, and the inertial matrix $[a_{mn}]$ is a function of the vector $\{\eta_n\}$. Invoking the quadratic truncation $a_{mn} = \delta_{mn} a_n + a_{lmn}\eta_l + \tfrac{1}{2} a_{jlmn}\eta_j \eta_l$ (which is consistent with the approximations in §3 below), we obtain

$$L = \tfrac{1}{2} a_n (\dot{\eta}_n^2 - \omega_n^2 \eta_n^2) - \tfrac{1}{2}\ddot{z}_0 \eta_n \eta_n + \tfrac{1}{2} a_{lmn} \eta_l \dot{\eta}_m \dot{\eta}_n + \tfrac{1}{4} a_{jlmn} \eta_j \eta_l \dot{\eta}_m \dot{\eta}_n, \tag{2.4}$$

where
$$a_n = k_n^{-1} \coth k_n d = g/\omega_n^2, \tag{2.5}$$

d is the ambient depth of the fluid, ω_n is the natural frequency of the nth mode (so that a_n is the length of an equivalent pendulum for that mode),

$$a_{lmn} = C_{lmn}\{1 + \tfrac{1}{2}(k_l^2 - k_m^2 - k_n^2)\, a_m a_n\}, \tag{2.6}$$

$$a_{1111} = \tfrac{1}{2} C_{11n}^2 k_n^4 a_1^2 a_n - \tfrac{2}{3} C_{1111} k_1^2 a_1 \tag{2.7}$$

(we anticipate that, of the a_{jlmn}, only a_{1111} is required), and

$$C_{lmn} = S^{-1} \iint \psi_l \psi_m \psi_n \,\mathrm{d}S, \quad C_{1111} = S^{-1} \iint \psi_1^4 \,\mathrm{d}S \tag{2.8a,b}$$

are pure numbers.

3. The average Lagrangian

We now invoke the arguments outlined in the fifth paragraph of §1 and recast (1.2) in the form

$$\eta_n = \delta_{1n} l\{p(\tau)\cos\omega t + q(\tau)\sin\omega t\} + \frac{l^2}{a_1}\{A_n(\tau)\cos 2\omega t + B_n(\tau)\sin 2\omega t + C_n(\tau)\}, \tag{3.1}$$

where $l = O(\epsilon^{\frac{1}{2}} a_1)$ is a lengthscale (see (3.7)),

$$\epsilon = \frac{a_0}{a_1} = \frac{\omega_1^2 a_0}{g} \quad (0 < \epsilon \ll 1) \tag{3.2}$$

is a scaling parameter (we may render $a_0 > 0$ by an appropriate choice of the origin of t), p, q, A_n, B_n and C_n are slowly varying, dimensionless amplitudes, and

$$\tau = \epsilon\omega t \tag{3.3}$$

is a slow time. We also introduce the frequency parameters

$$\beta = \frac{\omega^2 - \omega_1^2}{2\epsilon\omega_1^2} \approx \frac{\omega - \omega_1}{\epsilon\omega_1} \tag{3.4a}$$

and
$$\Omega_n = 4\left(\frac{\omega}{\omega_n}\right)^2 - 1 \approx \frac{4a_n - a_1}{a_1}. \tag{3.4b}$$

Substituting (1.1), (2.1) and (3.1) into (2.4), invoking (2.5) and (3.2)–(3.4), averaging over a 2π interval of ωt, and neglecting $O(\epsilon^3)$ on the hypotheses that $l = O(\epsilon^{\frac{1}{2}} a_1)$, $\beta = O(1)$ and $1/\Omega_n = O(1)$ ($\Omega_n = O(\epsilon)$ implies internal resonance, which is treated in §6), we obtain

$$\begin{aligned}\langle L \rangle = \tfrac{1}{2}\epsilon g l^2 \Big[& \dot{p}q - p\dot{q} + \beta(p^2+q^2) + p^2 - q^2 \\ & + \epsilon^{-1}\left(\frac{l}{a_1}\right)^2 \{\tfrac{1}{16} a_1 a_{1111}(p^2+q^2)^2 + \tfrac{1}{2}\Omega_n(A_n^2 + B_n^2) - C_n C_n \\ & + (a_{11n} - \tfrac{1}{4}a_{n11})[A_n(p^2-q^2) + 2B_n pq] + \tfrac{1}{2}a_{n11} C_n (p^2+q^2)\}\Big],\end{aligned} \tag{3.5}$$

wherein the dots now signify differentiation with respect to τ. It follows from Hamilton's principle that $\langle L \rangle$ must be stationary with respect to variations of each of p, q, A_n, B_n and C_n. Invoking this requirement for each of the secondary amplitudes, we obtain

$$(A_n, B_n) = -\Omega_n^{-1}(a_{11n} - \tfrac{1}{4}a_{n11})(p^2 - q^2, 2pq), \quad C_n = \tfrac{1}{4}a_{n11}(p^2+q^2). \tag{3.6a, b}$$

Substituting (3.6) into (3.5), invoking (2.5) and (3.4b), and choosing

$$l = 2\left(\frac{\epsilon}{|A|}\right)^{\frac{1}{2}} k_1^{-1} \tanh k_1 d, \tag{3.7}$$

where
$$A = \tfrac{1}{2}\{a_1 a_{1111} + a_{n11} a_{n11} - \tfrac{1}{2}a_1(4a_n - a_1)^{-1}(4a_{11n} - a_{n11})^2\} \tanh^4 k_1 d, \tag{3.8}$$

we obtain
$$\langle L \rangle = \epsilon g l^2 \{\tfrac{1}{2}(\dot{p}q - p\dot{q}) + H(p,q)\}\{1 + O(\epsilon)\}, \tag{3.9}$$

where
$$H = \tfrac{1}{2}(\beta+1)p^2 + \tfrac{1}{2}(\beta-1)q^2 + \tfrac{1}{4}(p^2+q^2)^2 \operatorname{sgn} A. \tag{3.10}$$

The preceding formulation may be reduced to that for weakly nonlinear free oscillations (I §6) by omitting the imposed acceleration $\ddot{z}_0$ in L (2.4) and the corresponding term $p^2 - q^2$ in $\langle L \rangle$ (3.5) and regarding ϵ as an arbitrary scaling parameter, which simplifications yield

$$\langle L \rangle = \tfrac{1}{2}\epsilon g l^2 \{\dot{p}q - p\dot{q} + \beta(p^2+q^2) + \tfrac{1}{2}(p^2+q^2)^2 \operatorname{sgn} A\}. \tag{3.11}$$

The frequency of the free oscillation described by (2.1) and (3.1) with p and q constant, as determined by requiring $\langle L \rangle$ (3.11) to be stationary with respect to p and q and invoking (3.4a) and (3.7), then is given by (cf. I(6.5))

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 - A\left(\frac{\overline{\eta_1^2}}{\lambda^2}\right), \tag{3.12}$$

where
$$\overline{\eta_1^2} = \tfrac{1}{2}l^2(p^2+q^2) \tag{3.13}$$

is the mean-square displacement of the primary mode,† and

$$\lambda = k_1^{-1} \tanh k_1 d \tag{3.14}$$

† The mean square over both space and time, as calculated from (2.1), (2.2c) and (3.1), is given by $\overline{\eta^2} = \overline{\eta_1^2}\{1 + O(\epsilon)\}$.

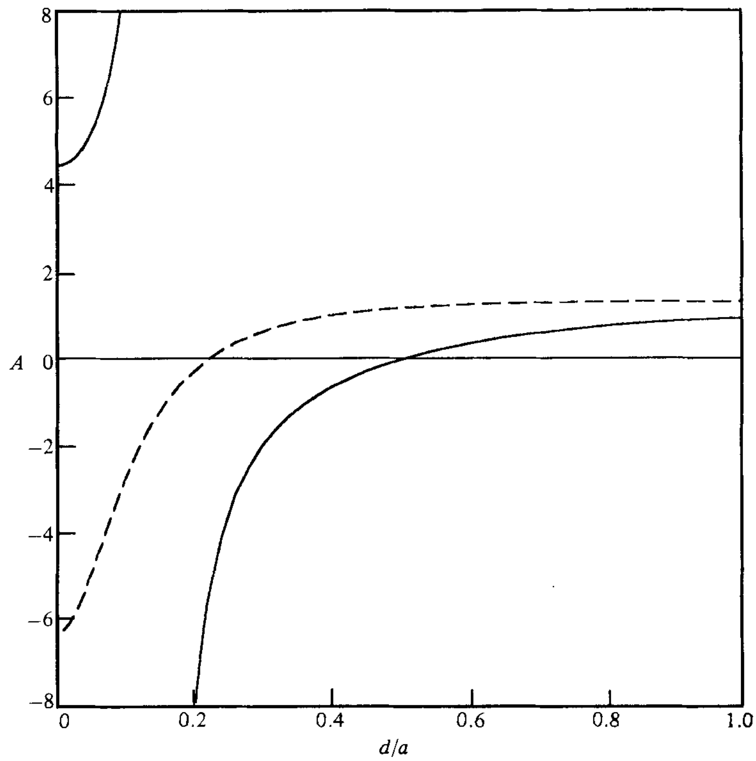

FIGURE 1. the parameters A_0 (– – –) and A_1 (——) for the dominant axisymmetric and antisymmetric modes in a circular cylinder of radius a and depth d. The singularity in A_1 at $d/a = 0.1523$ reflects the internal resonance (1.5). There are singularities in A_0 at $d/a = 0.1981$ due to the resonance (1.4) and at $d/a = 0.3470$ due to a resonance between the 01 and 04 modes, but these resonances are too narrow to be resolved on the scale of the present drawing.

is a reference length that reduces to d for shallow water and to $1/k_1$ for deep water. The calculation of the parameter A, which reduces to 1 for two-dimensional, deep-water waves (Rayleigh 1915), is pursued in Appendix B. Explicit results for the dominant axisymmetric (Mack 1962) and antisymmetric (Miles 1984) modes in a circular cylinder, say A_0 and A_1 respectively, are plotted in figure 1;

$$A_0 = (-6.303,\ \infty, 0, 1.307) \quad \text{at} \quad d/a = (0, 0.1981, 0.2230, \infty);$$

$$A_1 = (4.430, \infty, 0, 1.112) \quad \text{at} \quad d/a = (0, 0.1523, 0.5059, \infty).$$

The singularity at $d/a = 0.1981/0.1523$ reflects the internal resonance (1.4)/(1.5), the former of which is too narrow to be resolved in figure 1 (A_0 has a second, even narrower singularity at $d/a = 0.3470$ owing to the resonance between the 01 and 04 modes, for which $k_{01}a = 3.8317$ and $k_{04}a = 13.324$).

The choice (3.7) for l is inappropriate not only near $A = \infty$, but also near $A = 0$, where the higher-order terms incorporated in the present formulation make a null contribution to $\langle L \rangle$, and in the neighbourhood of which terms of the sixth order in the amplitude of η presumably would need to be retained to obtain a uniformly valid description of the nonlinearity.

4. Hamiltonian solutions

We now assume $A > 0$. If $A < 0$ it is necessary only to reverse the sign of β and interchange p and q throughout the subsequent development.

Requiring $\langle L \rangle$ to be stationary with respect to independent variations of p and q, we obtain the evolution equations†

$$\dot{p} = -\frac{\partial H}{\partial q} = -(\beta-1+p^2+q^2)\,q, \quad \dot{q} = \frac{\partial H}{\partial p} = (\beta+1+p^2+q^2)\,p, \qquad (4.1\,a,b)$$

in which H appears as a Hamiltonian and p and q are canonically conjugate variables. It follows that H is a constant of the motion, by virtue of which the solution may be reduced to quadrature. (In fact, the solution may be expressed in terms of elliptic integrals; however, this is of only passing interest in the present context.)

The fixed points (at which $\dot{p} = \dot{q} = 0$) of (4.1) are given by

$$p = q = 0, \qquad (4.2a)$$

$$p = 0, \quad q = \pm(1-\beta)^{\frac{1}{2}} \quad (\beta < 1), \qquad (4.2b)$$

$$p = \pm(-1-\beta)^{\frac{1}{2}}, \quad q = 0 \quad (\beta < -1). \qquad (4.2c)$$

A straightforward stability analysis (see §5) reveals that the fixed point at $p = q = 0$ is stable/unstable for $\beta^2 \gtrless 1$, those of (4.2b) are stable, and those of (4.2c) are unstable. (The stability criterion for the linear problem, which is governed by Mathieu's equation and corresponds to small perturbations about $p = q = 0$, is $\beta^2 > 1$ (Benjamin & Ursell 1954).) The stable/unstable fixed points are centres/saddle points. Summing up, we have the following configurations (see figure 2):

(a) $\beta > 1$, centre at $p = q = 0$;
(b) $-1 < \beta < 1$, saddle point at $p = q = 0$ and two centres at $p = 0$, $q = \pm(1-\beta)^{\frac{1}{2}}$;
(c) $\beta < -1$, three centres at $p = q = 0$ and $p = 0$, $q = \pm(1-\beta)^{\frac{1}{2}}$ and two saddle points at $p = \pm(-1-\beta)^{\frac{1}{2}}$, $q = 0$.

We remark that the motion relative to the basin vanishes at $p = q = 0$ and is harmonic (within $1+O(\epsilon^{\frac{1}{2}})$) at $p = 0$, $q = \pm(1-\beta)^{\frac{1}{2}}$. The mean-square displacement of the harmonic motion is given by $\frac{1}{2}l^2q^2 = \frac{1}{2}l^2(1-\beta)$. Invoking (3.7) for l, (3.4a) of β, and dividing by λ^2 (3.14), we obtain

$$\frac{\overline{\eta^2}}{\lambda^2} = \frac{2\epsilon}{A}(1-\beta) \qquad (4.3a)$$

$$= \left\{1-\left(\frac{\omega}{\omega_1}\right)^2+2\epsilon+O(\epsilon^2)\right\}\Big/A, \qquad (4.3b)$$

which is equivalent to (3.12) for $\epsilon = 0$ and manifests the non-uniform validity for our formulation near $A = 0$.

The phase-plane trajectories are given by $H =$ constant. Introducing the action-angle coordinates E and θ according to

$$p = (2E)^{\frac{1}{2}}\cos\theta, \quad q = (2E)^{\frac{1}{2}}\sin\theta \qquad (4.4a,b)$$

† The evolution equations (4.1) are isomorphic to (4.9) of Ockendon & Ockendon (1973); in particular, my β is related to their β according to $\beta_{\mathrm{M}} = -2\beta_{\mathrm{O}}$.

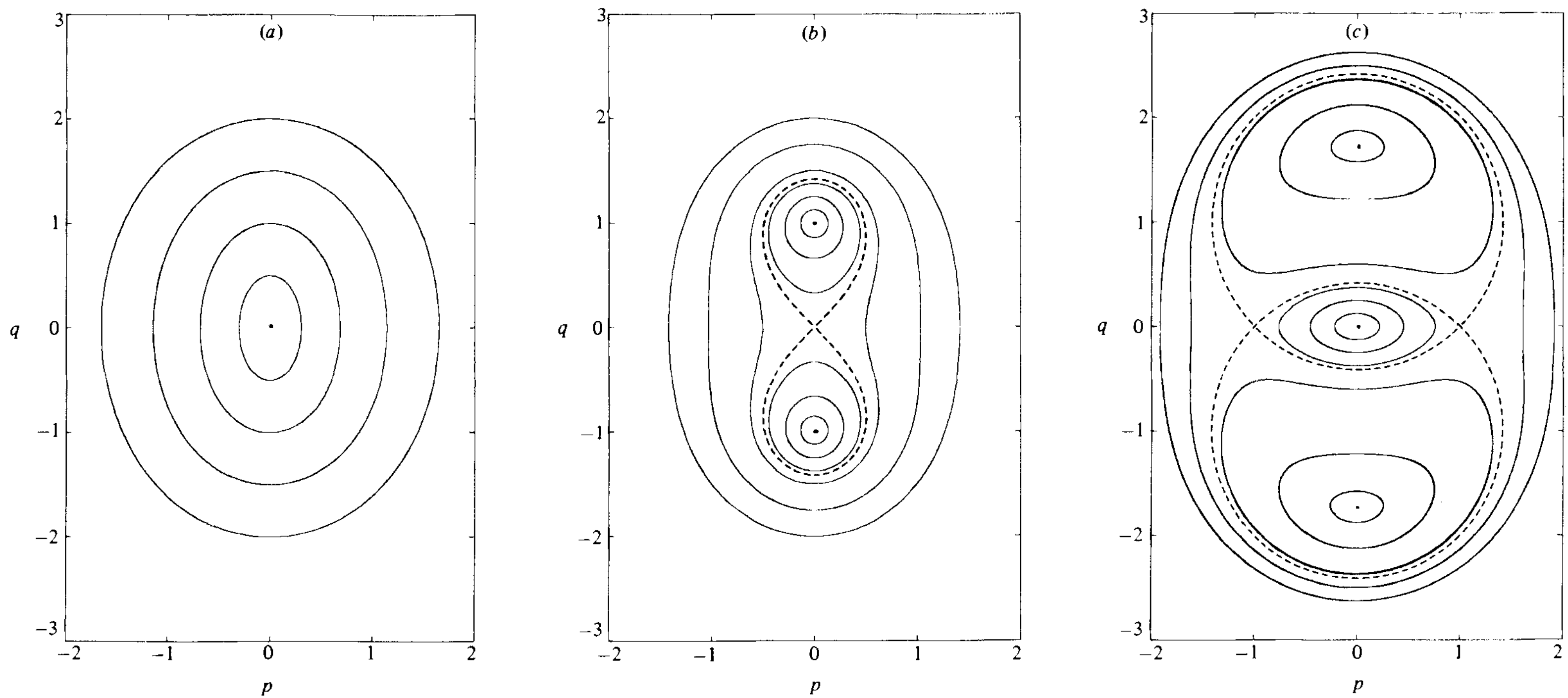

FIGURE 2. Fixed points and representative trajectories for the Hamiltonian system ($\alpha = 0$): (*a*) $\beta = 2$; (*b*) $\beta = 0$; (*c*) $\beta = -2$; which are representative of $\beta > 1$, $-1 < \beta < 1$, and $\beta < -1$ respectively. The separatrices are dashed.

in (3.10), we obtain

$$H = E^2 + E(\beta + \cos 2\theta), \tag{4.5}$$

$$\dot{E} = -\frac{\partial H}{\partial \theta} = 2E \sin 2\theta, \quad \dot{\theta} = \frac{\partial H}{\partial E} = 2E + \beta + \cos 2\theta. \tag{4.6a,b}$$

(*a*) The trajectories for $\beta > 1$ (figure 2*a*) constitute a nested set of closed loops about the centre at $p = q = 0$ ($H = 0$) with the family parameter H increasing monotonically outward across the set. It follows from (4.6*b*) that each of these trajectories is traversed in a counterclockwise sense (with increasing τ).

(*b*) The separatrix through the saddle point at $p = q = 0$ for $-1 < \beta < 1$ (figure 2*b*) is a vertical figure eight that intersects the q-axis at $q = \pm 2^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}$, encloses two symmetrically disposed, nested sets of closed trajectories about the centres at $p = 0$, $q = \pm(1-\beta)^{\frac{1}{2}}$, and is enclosed by a third nested set. H increases monotonically along radial lines from each of the centres, at which $H = -\frac{1}{4}(1-\beta)^2$, to the separatrix, on which $H = 0$, and then increases monotonically across the outer trajectories. It follows from (4.1) that each of the trajectories, including the upper and lower loops of the separatrix, is traversed in a counterclockwise sense.

(*c*) The separatrix through the saddle points at $p = \pm(-1-\beta)^{\frac{1}{2}}$, $q = 0$ for $\beta < -1$ (figure 2*c*) comprises a pair of closed, intersecting loops that intersect the q-axis at the four points $q = \pm(|\beta|^{\frac{1}{2}} \pm 1)$. The inner separatrix encloses a nested set of closed trajectories about the centre at $p = q = 0$. The outer separatrix encloses two nested sets about the outer centres at $p = 0$, $q = \pm(1-\beta)^{\frac{1}{2}}$ and is enclosed by a fourth nested set. H decreases monotonically outward from 0 at the inner centre to $-\frac{1}{4}(1+\beta)^2$ at the inner separatrix, increases monotonically outward along radial lines from $-\frac{1}{4}(1-\beta)^2$ at each of the outer centres to $-\frac{1}{4}(1+\beta)^2$ at the outer separatrix, and continues to increase monotonically across the outer trajectories. It follows from (4.1) that the inner separatrix and the trajectories about the inner centre are traversed in a clockwise sense, while the two loops of the outer separatrix, the trajectories about the outer centres, and the outer trajectories are traversed in a counterclockwise sense.

5. Damped solutions

The incorporation of weak, linear damping in the dynamical formulation is straightforward† and leads to the introducton of the terms $\alpha(p, q)$ on the left-hand sides of (4.1*a,b*), where

$$\alpha = \delta/\epsilon, \tag{5.1}$$

and δ is the ratio of actual to critical damping for free oscillations of the resonant mode. The resulting evolution equations are

$$\dot{p} + \alpha p + (\beta - 1 + p^2 + q^2)q = 0, \quad \dot{q} + \alpha q - (\beta + 1 + p^2 + q^2)p = 0. \tag{5.2a,b}$$

The fixed points of (5.2) are given by (cf. (4.2))

$$p = q = 0, \tag{5.3a}$$

$$p + \mathrm{i}q = \pm \mathrm{e}^{\mathrm{i}(\frac{1}{2}\pi - \phi)}(\gamma - \beta)^{\frac{1}{2}} \quad (\beta < \gamma), \tag{5.3b}$$

$$p + \mathrm{i}q = \pm \mathrm{e}^{\mathrm{i}\phi}(-\gamma - \beta)^{\frac{1}{2}} \quad (\beta < -\gamma), \tag{5.3c}$$

where

$$\cos 2\phi \equiv \gamma = (1 - \alpha^2)^{\frac{1}{2}} \quad (\alpha < 1); \tag{5.4}$$

† It suffices to substitute the dominant terms in (3.1) into the leading terms in the equation of motion for η_1, $\ddot{\eta}_1 + 2\delta\omega_1\dot{\eta}_1 + \omega_1^2\eta_1$, and compare the results with the linear terms in (4.1).

note that (5.3b) and (5.3c) differ only in the sign of γ. The origin is the only fixed point if $\alpha > 1$; conversely, $\epsilon > \delta$ is a necessary condition for subharmonic response.

The stability of a particular fixed point with respect to a small disturbance proportional to $\exp(\sigma\tau)$ is determined by the roots of the characteristic equation

$$\Delta = \begin{vmatrix} \sigma+\alpha+2pq & \beta-1+p^2+3q^2 \\ -(\beta+1+3p^2+q^2) & \sigma+\alpha-2pq \end{vmatrix} = 0. \tag{5.5}$$

Substituting $p = q = 0$ into (5.5), we obtain

$$\Delta = \sigma^2+2\alpha\sigma+\alpha^2+\beta^2-1, \tag{5.6a}$$

from which it follows that the fixed point at the origin is stable if and only if $\beta^2 > 1-\alpha^2$. Substituting (5.3b,c) into (5.5), we obtain

$$\Delta = \sigma^2+2\alpha\sigma+4\gamma(\gamma-\beta) \quad (\beta < \gamma), \tag{5.6b}$$

$$\Delta = \sigma^2+2\alpha\sigma+4\gamma(\gamma+\beta) \quad (\beta < -\gamma) \tag{5.6c}$$

respectively, from which it follows that the corresponding pairs of fixed points (5.3b,c) are stable/unstable. The stable/unstable fixed points are sinks/saddle points.

The Poincaré–Bendixson theorem implies that any solution of (5.2) must tend asymptotically to either a fixed point or a limit cycle. The logarithmic contraction rate for the area within a closed trajectory is given by (Lichtenberg & Lieberman 1983)

$$\frac{\partial\dot{p}}{\partial p}+\frac{\partial\dot{q}}{\partial q} = -2\alpha. \tag{5.7}$$

It follows from (5.7) that limit cycles are impossible and hence that every solution must tend to one of the stable fixed points for $\alpha > 0$.

Summing up, we have the following configurations for $0 < \alpha < 1$:

(a) $\beta > \gamma$, sink at $p = q = 0$;
(b) $-\gamma < \beta < \gamma$, saddle point at $p = q = 0$ and two sinks at (5.3b);
(c) $\beta < -\gamma$, three sinks at $p = q = 0$ and (5.3b), and two saddle points at (5.3c).

The only fixed point for $\alpha > 1$ is a sink at $p = q = 0$.

The trajectories for $\alpha > 0$, which may be determined either through the direct integration of (5.2) or through the integration of

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{\alpha p+(\beta-1+p^2+q^2)\,q}{\alpha q-(\beta+1+p^2+q^2)\,p} \tag{5.8}$$

or its action-angle (see (4.4)) equivalent,

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{2E(\sin 2\theta-\alpha)}{2E+\beta+\cos 2\theta}, \tag{5.9}$$

spiral inward to one of the sinks in each of configurations (a)–(c).

The total energy is given by

$$\hat{E} = \rho g S l^2 E\{1+O(\epsilon)\}, \tag{5.10}$$

where ρ is the fluid density. The action (which is a measure of the energy) at the stable fixed points is plotted as a function of the tuning parameter β in figure 3. If the motion is started from rest and β is increased through $-\gamma$, E may be expected to jump from 0 to γ and then to decrease linearly to 0 at $\beta = \gamma$ and remain at 0 as β increases above

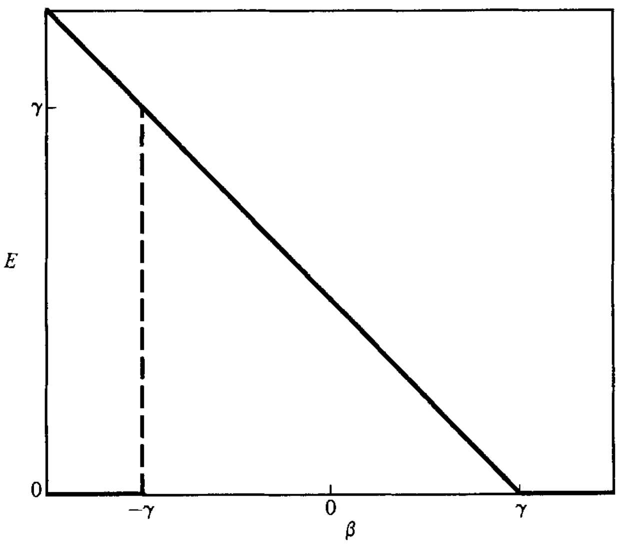

FIGURE 3. The dimensionless energy at the stable fixed points (5.3a,b). The dashed line indicates the jump from $E = 0$ as β is increased through $-\gamma$.

γ. If β is decreased through γ, E may be expected to increase linearly from 0 to some value in excess of γ for $\beta < -\gamma$ but eventually to drop to 0 at some lower value of β. If the motion is started from a state of finite energy with $\beta < -\gamma$, E may tend to $\frac{1}{2}(\gamma-\beta)$ rather than 0, depending on the initial conditions; if $\alpha \ll 1$ the dividing line for these initial conditions may be approximated by the inner separatrix for $\alpha = 0$:

$$2E = |\beta| - \cos 2\theta - 2(|\beta| - \cos^2\theta)^{\frac{1}{2}}|\sin\theta| \quad (\alpha \downarrow 0). \tag{5.11}$$

It would be desirable to confirm these predictions experimentally.

6. Internal resonance ($\omega_2 \approx 2\omega_1$)

We now assume that $\omega_2 - 2\omega_1 = O(\epsilon\omega)$, or, more precisely,

$$a_1(4a_2 - a_1)^{-1}(4a_{112} - a_{211})^2 = O(1/\epsilon) \tag{6.1}$$

for a particular pair of modes, $n = 1$ and 2. (Note that (6.1) sharply limits the bandwidth of most internal resonances; see below.) Referring to (3.7) and (3.8), we find that (6.1) implies $|A| = O(1/\epsilon)$ and $l = O(\epsilon)$. (Note that, as the limit $A \to \infty$ in (4.3) suggests, internal resonance *decreases* the amplitude of the subharmonic response *vis-à-vis* that of §3 for sufficiently small ϵ.) It follows that secondary modes are excited only with amplitudes $O(\epsilon^2)$; accordingly, we replace (3.1) by

$$\eta_n = l_n\{p_n(\tau)\cos n\omega t + q_n(\tau)\sin nwt\} \quad (n = 1, 2), \tag{6.2}$$

where $l_{1,2} = O(\epsilon)$ are lengthscales and ϵ and τ are given by (3.2) and (3.3). The corresponding approximation to the Lagrangian, obtained by substituting (1.1) into (2.4) and neglecting the quartic term (which now is small compared with the remaining terms), is

$$L = \tfrac{1}{2}\{a_n(\dot\eta_n^2 - \omega_n^2\eta_n^2 + 4\epsilon\omega_n^2\eta_n^2\cos 2\omega t) + a_{lmn}\eta_l\dot\eta_m\dot\eta_n\}, \tag{6.3}$$

wherein l, m, n are summed over 1, 2. Substituting (6.2) into (6.3), averaging over a 2π interval of ωt, introducing (cf. (3.4a))

$$\beta_n = \frac{n^2\omega^2-\omega_n^2}{2\epsilon n\omega^2} \approx \frac{n\omega-\omega_n}{\epsilon\omega} \quad (n = 1, 2), \tag{6.4}$$

and choosing (note that $\omega_2 \approx 2\omega_1$ and $\alpha_2 \approx \frac{1}{4}\alpha_1$ in the calculation of the higher-order terms)

$$l_2 = 2^{\frac{1}{2}} l_1 = \frac{4a_0}{4\alpha_{112}-\alpha_{211}}, \tag{6.5}$$

we obtain (cf. (3.9) and (3.10))

$$\langle L\rangle = \epsilon g l_1^2\{\tfrac{1}{2}(\dot{p}_n q_n - p_n \dot{q}_n) + H(p_1, q_1, p_2, q_2)\}, \tag{6.6}$$

where
$$H = \tfrac{1}{2}\beta_n(p_n^2+q_n^2)+\tfrac{1}{2}(p_1^2-q_1^2)(1+p_2)+p_1q_1q_2. \tag{6.7}$$

The total energy is given by (cf. (5.10))

$$\hat{E} = \tfrac{1}{2}\rho g S l_n^2(p_n^2+q_n^2)\{1+O(\epsilon)\}. \tag{6.8}$$

Substituting (2.6), together with $\epsilon = a_0/a_1$, into (6.5), and invoking (2.5) for α_1 and α_2, we obtain

$$l_2 = 4a_0(C_{112}[3+\{1-(k_2/k_1)^2\}\coth^2 k_1 d])^{-1}, \tag{6.9}$$

where C_{112} is given by (2.8a). Substituting k_1, k_2 and d/a from (1.4) and $C_{112} = -0.0106$ from table 1 (Appendix B) into (6.9), we obtain $l_2 = 32.2a_0$, which implies that the scaling (6.5) is inappropriate for this resonance and that proper scaling requires the retention of higher-order terms in the Lagrangian, as in §3. In contrast, the resonance between the dominant axisymmetric and antisymmetric modes in a circular cylinder, (1.5), for which $C_{112} = 0.410$ (Miles 1984), yields $l_2 = -0.237a_0$, which implies that the scaling (6.5) is appropriate.

The phase-plane equations implied by (6.6) and (6.7) are (cf. (4.1))

$$\dot{p}_1 = (1-\beta_1)q_1+p_2q_1-p_1q_2, \quad \dot{q}_1 = (1+\beta_1)p_1+p_1p_2+q_1q_2, \tag{6.10a,b}$$

$$\dot{p}_2 = -\beta_2 q_2 - p_1 q_1, \quad \dot{q}_2 = \beta_2 p_2 + \tfrac{1}{2}(p_1^2-q_1^2). \tag{6.10c,d}$$

Damping may be incorporated by adding $(\alpha_1 p_1, \alpha_1 q_1, \alpha_2 p_2, \alpha_2 q_2)$ to the left-hand side of (6.10a,b,c,d), where $\alpha_n = \delta_n/\epsilon$ and δ_n is the damping ratio for the nth mode.

We consider further the special case $\omega_2 = 2\omega_1$, for which

$$\beta_2 = 2\beta_1 \equiv 2\beta. \tag{6.11}$$

The fixed points of (6.10) then are given by

$$p_1 = p_2 = q_1 = q_2 = 0, \tag{6.12a}$$

$$p_1 = q_2 = 0, \quad q_1^2 = 4\beta(\beta-1), \quad p_2 = \beta-1 \quad (\beta > 1 \text{ or } \beta < 0), \tag{6.12b}$$

$$q_1 = q_2 = 0, \quad p_1^2 = 4\beta(\beta+1), \quad p_2 = -(\beta+1) \quad (\beta > 0 \text{ or } \beta < -1). \tag{6.12c}$$

The local stability of these fixed points is determined by the roots of the characteristic determinant

$$\Delta = \begin{vmatrix} \sigma+q_2 & -p_2+\beta-1 & q_1 & -p_1 \\ -p_2-\beta-1 & \sigma-q_2 & p_1 & q_1 \\ -q_1 & -p_1 & \sigma & 2\beta \\ p_1 & -q_1 & -2\beta & \sigma \end{vmatrix}. \tag{6.13}$$

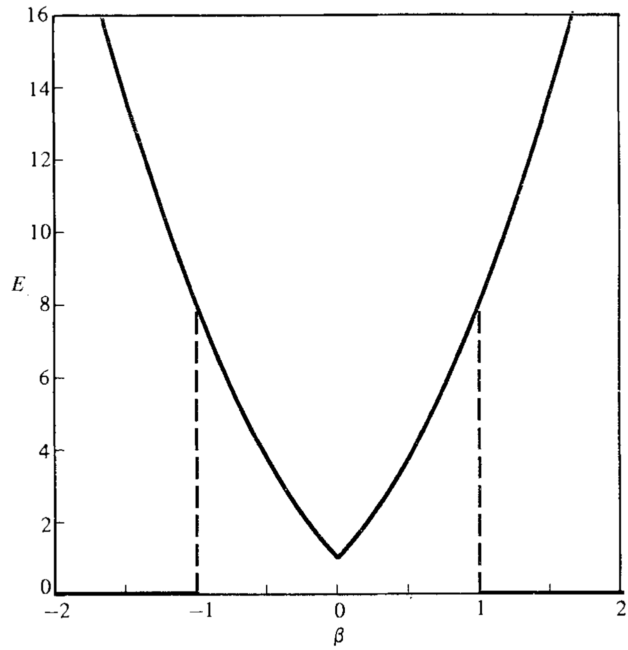

FIGURE 4. The dimensionless energy (6.15) at the stable fixed points (6.10a,b,c). The dashed lines indicate the jumps in E as β is either increased through -1 or decreased through $+1$.

Substituting (6.12a,b,c) into (6.13) in turn, we obtain

$$\Delta = (\sigma^2+\beta^2-1)(\sigma^2+4\beta^2), \tag{6.14a}$$

$$\Delta = \sigma^4+4\beta(3\beta-2)\sigma^2+16\beta^2(1-\beta), \tag{6.14b}$$

$$\Delta = \sigma^4+4\beta(3\beta+2)\sigma^2+16\beta^2(\beta+1), \tag{6.14c}$$

from which it follows that the fixed points (6.12a,b,c) are stable/unstable for: (a) $\beta^2 \gtrless 1$, (b) $\beta < 0/> 1$, (c) $\beta > 0/< -1$. Summing up, we have the following configurations:

(i) $\beta > 1$, three centres (6.12a,c) and two saddle points (6.12b);
(ii) $0 < \beta < 1$, a saddle point (6.12a) and two centres (6.12c);
(iii) $-1 < \beta < 0$, a saddle point (6.12a) and two centres (6.12b);
(iv) $\beta < -1$, three centres (6.12a,b) and two saddle points (6.12c).

The bifurcations at $\beta = 0$ and ± 1 are of Poincare's type, at which σ goes through zero. There are no Hopf-bifurcation points (at which the real part of a pair of complex-conjugate zeros of Δ vanishes and at which bifurcation to a limit cycle may occur), which suggests that, in the presence of finite but weak damping (centers $\rightarrow$ sinks), the solution will terminate at one of the sinks (which one will depend on the initial conditions). We remark that some finite-amplitude harmonic motion is possible (although it may be difficult to attain) for every value of β, in contrast to the results in §§4 and 5, where $\beta < \gamma$ is necessary for such a motion.

The dimensionless energy

$$E = \tfrac{1}{2}(p_1^2+q_1^2)+(p_2^2+q_2^2) \tag{6.15}$$

at the stable fixed points is plotted as a function of the tuning parameter β in figure 4. There are now two jumps (at $\beta = \pm 1$) in the equilibrium energy, in contrast to the single jump in figure 3; moreover, the energy decreases as β decreases below (increases above) $\beta = 1$ (-1) and has a minimum at $\beta = 0$. Here again, experimental confirmation would be desirable.

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Appendix A. Parametrically excited pendulum

The point of suspension of a simple pendulum of mass m and length l is subjected to the vertical displacement

$$z_0(t) = \epsilon l \cos 2\omega t \quad (0 < \epsilon \ll 1). \tag{A 1}$$

The corresponding Lagrangian is

$$L = \tfrac{1}{2}ml^2\dot{\theta}^2 - m(g+\ddot{z}_0)\,l(1-\cos\theta), \tag{A 2}$$

where θ is the angular displacement from the stable equilibrium position (in which the bob is directly below the point of suspension).

A suitable form for the solution of the equation of motion implied by (A 2) in the limit $\epsilon \downarrow 0$ with $\omega^2 - \omega_0^2 = O(\epsilon\omega^2)$, where $\omega_0^2 = g/l$, is

$$\theta = 4\epsilon^{\frac{1}{2}}\{p(\tau)\cos\omega t + q(\tau)\sin\omega t\}, \quad \tau = \epsilon\omega t. \tag{A 3a,b}$$

Substituting (A 3) into (A 2), approximating $1-\cos\theta$ by $\theta^2/2-\theta^4/4!$, averaging L over a 2π interval of ωt, and introducing

$$\beta = \frac{\omega^2-\omega_0^2}{2\epsilon\omega_0^2}, \tag{A 4}$$

we obtain

$$\langle L\rangle = 8\epsilon^2 mgl\{\dot{p}q - p\dot{q} + (\beta+1)\,p^2 + (\beta-1)\,q^2 + \tfrac{1}{2}(p^2+q^2)^2\} + O(\epsilon^3), \tag{A 5}$$

which is isomorphic to (3.9) for $A > 0$.

Appendix B. Calculation of A

The calculation of A from (3.8) requires the calculation of a_{11n}, a_{n11} and a_{1111} from the eigenfunctions and eigenvalues for a particular set of modes. Combining (2.5)–(2.7) and (3.4b) in (3.8), we obtain

$$A = \tfrac{1}{4}C_{11n}^2\,F_n(k_1 d) - \tfrac{1}{3}C_{1111}\,T_1^2, \tag{B 1}$$

wherein n is summed over all modes for which $C_{11n} \neq 0$,

$$F_n = \hat{k}_n^3\frac{T_1}{T_n} + 2(T_1^2 - 1 + \tfrac{1}{2}\hat{k}_n^2)^2 - \left\{4\hat{k}_n^{-1}\frac{T_1}{T_n} - 1\right\}^{-1}\left\{3T_1^2 + 1 - \tfrac{1}{2}\hat{k}_n^2 - 2\hat{k}_n\frac{T_1}{T_n}\right\}^2, \tag{B 2}$$

$$\hat{k}_n = k_n/k_1, \quad T_n = \tanh k_n d. \tag{B 3a,b}$$

The normalized axisymmetric modes in a circular cylinder of radius a and depth d are described by

$$\psi_n = \frac{J_0(k_n r)}{J_0(\kappa_n)}, \quad J_1(\kappa_n) = 0 \quad (\kappa_n \equiv k_n a,\ n = 1,2,\ldots). \tag{B 4a,b}$$

n	κ_n	C_{11n}
1	3.8317	−0.87465
2	7.0156	0.88675
3	10.1735	−0.01058
4	13.3237	−0.00114

TABLE 1. The axisymmetric eigenvalues and correlation integrals

If the primary mode corresponds to $\kappa_1 = 3.8317$, the non-axisymmetric modes (which exhibit the azimuthal variations $\cos m\theta$ and $\sin m\theta$ in the polar coordinates r and θ) are orthogonal to both ψ_1 and ψ_1^2, in consequence of which they do not enter the present calculation, and the required integrals are

$$C_{11n} = \frac{2}{J_0^2(\kappa_1)\,J_0(\kappa_n)}\int_0^1 J_0^2(\kappa_1 x)\,J_0(\kappa_n x)\,x\,\mathrm{d}x, \quad C_{1111} = \frac{2}{J_0^4(\kappa_1)}\int_0^1 J_0^4(\kappa_1 x)\,x\,\mathrm{d}x. \tag{B 5a,b}$$

Substituting the numerical values of C_{11n} (table 1) and $C_{1111} = 2.5515$, together with the corresponding eigenvalues, into (B 1) and (B 2), we obtain the result plotted in figure 1, which agrees (within the accuracy of his graphs) with that obtained by Mack (1962), in whose notation

$$A = -\tfrac{1}{2}G_c/J_0^2(K_1) \quad (K_1 \equiv \kappa_1). \tag{B 6}$$

The dominant mode in a circular cylinder is described by

$$\psi_1 = \kappa_1\left(\frac{2}{\kappa_1^2-1}\right)^{\frac{1}{2}}\frac{J_1(k_1 r)}{J_1(\kappa_1)}\cos\theta, \tag{B 7}$$

where $\kappa_1 = 1.8412$. This mode couples with (i.e. $C_{11n} \neq 0$ for) the axisymmetric and $\cos 2\theta$ modes, and the summations in (3.8) are over the radial wavenumbers of these two sets of modes. The required calculations have been carried out elsewhere (Miles 1984) and yield the result plotted in figure 1.

Appendix C. Internal resonance ($\omega_2 \approx \omega_1$)

We now assume that $\omega_2 - \omega_1 = O(\epsilon\omega)$ for a particular pair of modes, $n = 1$ and 2, and replace (3.1) and (3.4a) by

$$\eta_n = (\delta_{1n}+\delta_{2n})\,\hat{l}\{p_n(\tau)\cos\omega t + q_n(\tau)\sin\omega t\} + \frac{\hat{l}^2}{a_1}\{A_n(\tau)\cos 2\omega t + B_n(\tau)\sin 2\omega t + C_n(\tau)\} \tag{C 1}$$

and

$$\beta_n = \frac{\omega^2-\omega_n^2}{2\epsilon\omega^2} \approx \frac{\omega-\omega_n}{\epsilon\omega}, \tag{C 2}$$

where $\hat{l}$ is a lengthscale (to be determined), and ϵ and τ are given by (3.2) and (3.3). Substituting (C 1) into (2.3) and proceeding as in §3, we obtain (note that $\omega_2 \approx \omega_1$ implies $a_2 \approx a_1$)

$$\langle L\rangle = \tfrac{1}{2}\epsilon g l^2\Big[\dot{p}_n q_n - p_n\dot{q}_n + \beta_n(p_n^2+q_n^2) + p_n p_n - q_n q_n + \epsilon^{-1}\Big(\frac{\hat{l}}{a_1}\Big)^2\{\tfrac{1}{2}\Omega_n(A_n^2+B_n^2) - C_n C_n$$
$$+ (a_{mnl} - \tfrac{1}{4}a_{lmn})(A_l(p_m p_n - q_m q_n) + B_l(p_m q_n + p_n q_m)) + \tfrac{1}{2}a_{lmn}C_l(p_m p_n + q_m q_n)$$
$$+ \tfrac{1}{32}a_1(a_{jlmn} + a_{jlnm})((p_j p_l + q_j q_l)(p_m p_n + q_m q_n) + 2(p_l q_m - p_m q_l)(p_j q_n - p_n q_j))\}\Big], \tag{C 3}$$

wherein $p_n, q_n = 0$ unless $n = 1$ or 2. Requiring $\langle L\rangle$ to be stationary with respect to each of the secondary amplitudes, we obtain

$$(A_l, B_l) = -\Omega_l^{-1}(a_{mnl} - \tfrac{1}{4}a_{lmn})(p_m p_n - q_m q_n, p_m q_n + p_n q_m), \tag{C 4a}$$

$$C_l = \tfrac{1}{4}a_{lmn}(p_m p_n + q_m q_n), \tag{C 4b}$$

the substitution of which into (C 3) yields $\langle L\rangle$ in the form (cf. (3.9))

$$\langle L\rangle = \epsilon g l^2\{\tfrac{1}{2}(\dot{p}_n q_n - p_n\dot{q}_n) + H(p_1, q_1, p_2, q_2)\}, \tag{C 5}$$

with

$$H = \tfrac{1}{2}\beta_n(p_n^2+q_n^2) + \tfrac{1}{2}(p_n p_n - q_n q_n)$$
$$+ \tfrac{1}{2}\epsilon^{-1}\Big(\frac{\hat{l}}{a_1}\Big)^2[-\tfrac{1}{2}\Omega_n(A_n^2+B_n^2) + C_n C_n + \tfrac{1}{64}a_1(a_{jlmn} + a_{jlnm})$$
$$\times\{(p_j p_l + q_j q_l)(p_m p_n + q_m q_n) + 2(p_l q_m - p_m q_l)(p_j q_n - p_n q_j)\}], \tag{C 6}$$

wherein A_l, B_l and C_l are given by (C 4).

More explicit results require the assumption of specific modes, and the algebra for any nontrivial case appears to be formidable. The simplest case is that in which ψ_1 and ψ_2 differ only in an azimuthal phase difference of $\frac{1}{2}\pi$ (as in Miles 1984), but this reduces to the case of a single primary mode if the angular momentum is zero (as it is for vertical translation of a cylinder after any initial angular momentum has been dissipated).

Appendix D. Capillary effects

see corrigendum Vol 154 p 535

The capillary energy due to a uniform surface tension $\rho\hat{T}$ is given by

$$V = \tfrac{1}{2}\rho\hat{T}\iint_S (\nabla\eta)^2\,\mathrm{d}S, \tag{D 1}$$

where the integral is over the free surface. Substituting (2.1) into (D 1) and invoking

$$\iint \nabla\psi_m\cdot\nabla\psi_n\,\mathrm{d}S = \delta_{mn}k_n^2 S, \tag{D 2}$$

which follows from (2.2), we obtain

$$V = \tfrac{1}{2}\rho S\hat{T}k_n^2\eta_n^2. \tag{D 3}$$

Substituting (3.1) into (D 3) and averaging over ωt, we obtain

$$\langle V\rangle = \tfrac{1}{4}\rho S\hat{T}l^2\Big[k_1^2(p^2+q^2) + \Big(\frac{l}{a_1}\Big)^2 k_n^2(A_n^2+B_n^2+2C_n^2)\Big]. \tag{D 4}$$

Dividing $\langle V \rangle$ by ρS (to allow for the normalization of L in (2.3)) and subtracting the result from (3.5), we find that the following changes must be made in (3.5) to incorporate capillary effects: replace (3.4*a*,*b*) by

$$\beta_* = (2\epsilon)^{-1}\left[\left(\frac{\omega}{\omega_1}\right)^2 - 1 - k_1^2 l_*^2\right] \tag{D 5a}$$

and
$$\Omega_{n*} = 4\frac{a_n}{a_1} - 1 - k_n^2 l_*^2 \tag{D 5b}$$

and multiply $C_n C_n$ by $1 + k_n^2 l_*^2$, where

$$l_* \equiv (\hat{T}/g)^{\frac{1}{2}} \tag{D 6}$$

is the capillary length and is approximately 2.8 mm for clean water. It then follows that Ω_n must be replaced by Ω_{n*} in (3.6*a*), the right-hand side of (3.6*b*) must be divided by $1 + k_n^2 l_*^2$, and (3.8) must be replaced by

$$A = \frac{1}{2}\left\{a_1 a_{1111} + (1 + k_n^2 l_*^2)^{-1} a_{n11} a_{n11} - \frac{1}{2}\left[4\frac{a_n}{a_1} - 1 - k_n^2 l_*^2\right]^{-1} (4a_{11n} - a_{n11})^2\right\} \tanh^4 k_1 d. \tag{D 7}$$

Appendix E. A note on Dodge, Kana & Abramson (1965)

Dodge *et al.* (1965) start from a normal-mode expansion for a circular basin that is equivalent to (2.1). They then assume that the primary mode is the dominant mode, as described by (B 7) above, and deduce that only those secondary modes with azimuthal wavenumbers 0 and 2 are excited by the lowest-order (quadratic) nonlinear terms. They also argue, on numerical grounds, that only the lowest (wavenumbers k_{01} and k_{21}) secondary modes need be retained, thereby reducing their system to three degrees of freedom. Starting from Laplace's equation for the velocity potential, the kinematic boundary conditions on the walls, and the (nonlinear) kinematic and dynamic boundary conditions at the free surface and retaining terms of first, second and third order in the amplitude of the primary mode, they arrive at the three equations of motion (their (20) and (21))

$$\begin{aligned}\ddot{a}_1 + (1 - 4\sigma^2\epsilon \cos 2\sigma t)\, a_1(1 + K_1 a_1^2 + K_0 a_0 - K_2 a_2) + 0.034\,780\lambda_1^2 \ddot{a}_1 a_1^2 \\ + k_1 \dot{a}_1^2 a_1 + 0.165\,118\ddot{a}_0 a_1 - 0.198\,686\ddot{a}_2 a_1 + k_0 \dot{a}_0 \dot{a}_1 - k_2 \dot{a}_2 \dot{a}_1 = 0,\end{aligned} \tag{E 1}$$

$$\begin{aligned}\ddot{a}_0 + \lambda_0 \tanh \lambda_0 b(1 - 4\sigma^2\epsilon \cos 2\sigma t)\, a_0 \\ - \ddot{a}_1 a_1(0.121\,482\lambda_0 \tanh \lambda_0 b - 0.263\,074\lambda_1^2) \\ + \dot{a}_1^2 [\lambda_0^2 \tanh \lambda_0 b(0.070\,796\lambda_1^2 - 0.060\,741) + 0.263\,074\lambda_1^2] = 0,\end{aligned} \tag{E 2}$$

$$\begin{aligned}\ddot{a}_2 + \lambda_2 \tanh \lambda_2 b(1 - 4\sigma^2\epsilon \cos 2\sigma t)\, a_2 \\ + \ddot{a}_1 a_1(0.350\,807\lambda_2 \tanh \lambda_2 b - 0.482\,670\lambda_1^2) \\ + \dot{a}_1^2 [\lambda_2 \tanh \lambda_2 b(0.175\,403 - 0.065\,931\lambda_1^2) - 0.482\,670\lambda_1^2] = 0\end{aligned} \tag{E 3}$$

(in their notation except that I have dropped the redundant second subscript on all symbols).

These equations cannot be derived from a Lagrangian of the form (2.3) above, which should be possible since they presumably are derived from the same basic equations through the expansion of the free-surface displacements in the same set of orthogonal modes. An immediate difficulty is posed by the terms in a_1^3, $a_1 a_0$ and $a_1 a_2$ in (E 1), none of which should appear in equations of motion derived from a

Lagrangian in which the potential energy is in normal form (contains only squares of the generalized coordinates); however, this difficulty can be circumvented by dividing (E 1) through by $1+K_1 a_1^2+K_0 a_0-K_2 a_2$, and approximating this divisor by 1 except in the coefficient of $\ddot{a}_1$, where its inverse may be approximated by $1-K_1 a_1^2-K_0 a_0+K_2 a_2$. After making this change, the coefficients of $a_1^2\ddot{a}_1$ and $a_1\dot{a}_1^2$ should be equal, as also should the coefficients of $\ddot{a}_1 a_0$ and $\dot{a}_1\dot{a}_0$ and of $\ddot{a}_1 a_2$ and $\dot{a}_1\dot{a}_2$; in fact, none of these equalities is satisfied (the paired coefficients differ in form, so that the difficulty goes beyond numerical inequality).

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